A CLASS OF GORENSTEIN KOSZUL ALGEBRAS

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ABSTRACT. We show that the class of artinian Gorenstein local rings defined in [2] are Koszul algebras.

Let k be a field not algebraic over a finite field. In this note we prove via Buchberger's Algorithm that the class of ideals defining the artinian Gorenstein local k-algebras of [2] are generated by a Gröbner basis of homogeneous quadratics. It follows (from [1], for example) that the rings are Koszul k-algebras.

Let $Q = k[T, U, V, X, Y, Z]$. The ideals I of Q defining the rings R from [2] are generated by the following fifteen quadratic relations:

$$Z^2,\ UZ - TX - \alpha UV,\ U^2,\ YZ + VY,\ UY,\ Y^2 - TX - (\alpha - 1)UV,$$
$$XZ + \alpha VX,\ UX,\ XY,\ X^2 - TX - TV,\ TZ + TY + \alpha VX,\ TU,$$
$$TY - VX + TV,\ T^2 + (\alpha + 1)UV - VY,\ V^2,$$

where $\alpha \in k$ is an element of infinite multiplicative order. We show that in the variable ordering $Z > U > Y > X > T > V$, the generators for I listed above are a Gröbner basis for their ideal with respect to the reverse lexicographic monomial ordering. This is the case independent of which α is chosen. Note that the ordering of the monomial quadratics is as follows: $Z^2 > UZ > U^2 > YZ > UY > Y^2 > XZ > UX > XY > X^2 > TZ > TU > TY > TX > T^2 > VZ > UV > VY > VX > TV > V^2$. Therefore we have listed the generators of I above in decreasing order on initial terms, and each generator has its monomial terms listed in decreasing order.

Let $g_1, \dots, g_{15}$ denote the fifteen generators of I listed in the order in which they appear above. We will use Buchberger's Criterion to show that these generators of I are a Gröbner basis for their ideal: Let l_i denote the initial term of g_i. and set $m_{ij} = l_i/gcd(l_i, l_j)$ and define $h_{ij} \in A$ by the standard expression

$$m_{ji}g_i - m_{ij}g_j = \sum_r c_r^{(ij)} g_r + h_{ij}$$

where $c_r^{(ij)} \in Q$, the initial term of $c_r^{(ij)} g_r$ is less than or equal to the initial term of $g_i m_{ji} - g_j m_{ij}$ for all r, and none of the monomials of h_{ij} is in the ideal $(l_1, \dots, l_{15})$. Then Buchberger's Criterion states that $g_1, \dots, g_{15}$ are a Gröbner basis for their ideal if and only if $h_{ij} = 0$ for all $1 \le i, j \le 15$. This is what we use to establish our claim.

A priori one would need to compute $\binom{15}{2} = 105$ of the h_{ij}. (Of course, $h_{ij} = -h_{ji}$, and $h_{ii} = 0$.) However, it is trivial that if g_i and g_j are monomials, then $h_{ij} = 0$. Since there are seven monomial generators among $g_1, \dots, g_{15}$, this accounts for

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$\binom{7}{2} = 21$ of the computations. Furthermore, if g_i is a monomial and $\gcd(g_i, g_j) = 1$, then obviously $h_{ij} = 0$. Since

$$\begin{aligned}
&\gcd(g_1, g_6) = \gcd(g_1, g_{10}) = \gcd(g_1, g_{13}) = \gcd(g_1, g_{14}) = \gcd(g_3, g_4) = \\
&\gcd(g_3, g_6) = \gcd(g_3, g_7) = \gcd(g_3, g_{10}) = \gcd(g_3, g_{11}) = \gcd(g_3, g_{13}) = \\
&\gcd(g_3, g_{14}) = \gcd(g_5, g_7) = \gcd(g_5, g_{10}) = \gcd(g_5, g_{11}) = \gcd(g_5, g_{14}) = \\
&\gcd(g_8, g_4) = \gcd(g_8, g_6) = \gcd(g_8, g_{11}) = \gcd(g_8, g_{13}) = \gcd(g_8, g_{14}) = \\
&\gcd(g_9, g_2) = \gcd(g_9, g_{11}) = \gcd(g_9, g_{14}) = \gcd(g_{12}, g_4) = \gcd(g_{12}, g_6) = \\
&\gcd(g_{12}, g_7) = \gcd(g_{12}, g_{10}) = \gcd(g_{15}, g_2) = \gcd(g_{15}, g_4) = \gcd(g_{15}, g_6) = \\
&\gcd(g_{15}, g_7) = \gcd(g_{15}, g_{10}) = \gcd(g_{15}, g_{11}) = (g_{15}, g_{13}) = \gcd(g_{15}, g_{14}) = 1,
\end{aligned}$$

this accounts for 35 more computations. Thus there are only 49 remaining computations to exhibit. We do this in decreasing order on initial terms of the first generator of the pair. When a standard expression of an element E has appeared previously, we use the notation $(E)_n$ to indicate which line:

1. $(U)g_1 - (Z)g_2 = [TXZ + \alpha UVZ] = [(\alpha V)g_2 + (T)g_7 + (\alpha^2 U)g_{15}]$
2. $(Y)g_1 - (Z)g_4 = -[VYZ] = -[(V)g_4 + (-Y)g_{15}]$
3. $(X)g_1 - (Z)g_7 = -\alpha[VXZ] = -\alpha[(V)g_7 + (-\alpha X)g_{15}]$
4. $(T)g_1 - (Z)g_{11} = -[TYZ] - \alpha[VXZ] = -[(T)g_4 + (-V)g_{13} + (T - X)g_{15}] - \alpha(VXZ)_3$

This takes care of terms involving g_1.

5. $(U)g_2 - (Z)g_3 = -[TUX] - \alpha[U^2V] = -[(T)g_8] - \alpha[(V)g_3]$
6. $(Y)g_2 - (U)g_4 = -[TXY] - (\alpha + 1)[UVY] = -[(T)g_9] - (\alpha + 1)[(V)g_5]$
7. $(Y)g_2 - (Z)g_5 = -[TXY] - \alpha[UVY] = -(TXY)_6 - \alpha(UVY)_6$
8. $(Y^2)g_2 - (UZ)g_6 = [TXY^2] + \alpha[UVY^2] + [TUXZ] + (\alpha - 1)[U^2VZ] = [(TY)g_9] + \alpha[(VY)g_5] + [(TZ)g_8] + (\alpha - 1)[(VZ)g_3]$
9. $(X)g_2 - (U)g_7 = -[TX^2] - \alpha[UVX] - \alpha[UVX] = -[(\alpha+1)(-V)g_8 + (V)g_9 + (T)g_{10} + (V + X)g_{14} + (Y - (\alpha + 1)U)g_{15}] - \alpha[(V)g_8] - \alpha[(V)g_8]$
10. $(X)g_2 - (Z)g_8 = -[TX^2] - \alpha[UVX] = -(TX^2)_9 - \alpha(UVX)_9$
11. $(X^2)g_2 - (UZ)g_{10} = -[TX^3] - \alpha[UVX^2] + [TUXZ] + [TUVZ] = -[(-(\alpha + 1)(VX + V^2))g_8 + (VX + V^2)g_9 + (TX)g_{10} + (X^2 + VX)g_{14}] - \alpha[(VX)g_8] + (TUXZ)_8 + [(VZ)g_{12}]$
12. $(T)g_2 - (U)g_{11} = -[T^2X] - \alpha[TUV] - [TUY] - \alpha[UVX] = -[(\alpha+1)(-V)g_8 + (V)g_9 + (X)g_{14}] - \alpha[(V)g_{12}] - [(Y)g_{12}] - \alpha(UVX)_9$
13. $(T)g_2 - (Z)g_{12} = -[T^2X] - \alpha[TUV] = -(T^2X)_{12} - \alpha(TUV)_{12}$
14. $(TY)g_2 - (UZ)g_{13} = -[T^2XY] - \alpha[TUVY] + [UVXZ] - [TUVZ] = -[(T^2)g_9] - \alpha[(TV)g_5] + [(VZ)g_8] - (TUVZ)_{11}$
15. $(T^2)g_2 - (UZ)g_{14} = -[T^3X] - \alpha[T^2UV] - (\alpha + 1)[U^2VZ] + [UVYZ] = -[(TV)g_9 + (\alpha + 1)(-VX)g_{12} + (TX)g_{14}] - \alpha[(TV)g_{12}] - (\alpha + 1)[(VZ)g_3] + [(VZ)g_5]$

This takes care of terms involving g_2 and g_3.

16. $(U)g_4 - (Z)g_5 = [UVY] = (UVY)_6$
17. $(Y)g_4 - (Z)g_6 = VY^2 + TXZ + (\alpha - 1)UVZ = [TXZ + \alpha UVZ] + [VY^2 - UVZ] = (TXZ + \alpha UVZ)_1 + [(-V)g_2 + (V)g_6 + (-U)g_{15}]$
18. $(X)g_4 - (Y)g_7 = (1 - \alpha)[VXY] = (1 - \alpha)[(V)g_9]$
19. $(X)g_4 - (Z)g_9 = [VXY] = (VXY)_{18}$

20. $(X^2)g_4-(YZ)g_{10}=[VX^2Y]+[TXYZ]+[TVYZ]=[(VX)g_9]+[(TZ)g_9]+[(TV)g_4+(-TY)g_{15}]$
21. $(T)g_4-(Y)g_{11}=[TVY]-[TY^2]-\alpha[VXY]=[(V)g_{13}+(X-T)g_{15}]-[(V)g_9+(Y-V)g_{13}+(T-X)g_{15}]-\alpha(VXY)_{18}$
22. $(T)g_4-(Z)g_{13}=[TVY]+[VXZ]-[TVZ]=(TVY)_{21}+(VXZ)_3-[(V)g_{11}+(-V)g_{13}+(T-2X)g_{15}]$
23. $(T^2)g_4-(YZ)g_{14}=[T^2VY]-(\alpha+1)[UVYZ]+[VY^2Z]=[(VY)g_{14}+(Y^2-(\alpha+1)UY)g_{15}]-(\alpha+1)(UVYZ)_{15}+[(VY)g_4+(-Y^2)g_{15}]$

This takes care of terms involving g_4.

24. $(Y)g_5-(U)g_6=[TUX]+(\alpha-1)[U^2V]=(TUX)_5+(\alpha-1)(U^2V)_5$
25. $(T)g_5-(U)g_{13}=[UVX]-[TUV]=(UVX)_9-(TUV)_{12}$

This takes care of terms involving g_5.

26. $(XZ)g_6-(Y^2)g_7=[TX^2Z]-(\alpha-1)[UVXZ]-\alpha[VXY^2]=[(TX)g_7+(-\alpha TV)g_{10}+(-(\alpha^2+\alpha)UX-\alpha XY-\alpha T^2)g_{15}]-(\alpha-1)(UVXZ)_{14}-\alpha[(VY)g_9]$
27. $(X)g_6-(Y)g_9=-[TX^2]-(\alpha-1)[UVX]=-(TX^2)_9-(\alpha-1)(UVX)_9$
28. $(X^2)g_6-(Y^2)g_{10}=-[TX^3]-(\alpha-1)[UVX^2]+[TXY^2]+[TVY^2]=-(TX^3)_{11}-(\alpha-1)(UVX^2)_{11}+(TXY^2)_8+[(TV)g_6+(\alpha+1)(-V^2)g_8+(V^2)g_9+(VX)g_{14}+(\alpha-1)(TU)g_{15}]$
29. $(TZ)g_6-(Y^2)g_{11}=-[T^2XZ]-(\alpha-1)[TUVZ]-[TY^3]-\alpha[VXY^2]=-[(\alpha+1)(-VZ)g_8+(VZ)g_9+(XZ)g_{14}]-(\alpha-1)(TUVZ)_{11}-[(TY)g_6+(T^2)g_9+(\alpha-1)(VY)g_{12}]-\alpha(VXY^2)_{26}$
30. $(T)g_6-(Y)g_{13}=-[T^2X]-(\alpha-1)[TUV]+[VXY]-[TVY]=-(T^2X)_{12}-(\alpha-1)(TUV)_{12}+(VXY)_{18}-(TVY)_{21}$
31. $(T^2)g_6-(Y^2)g_{14}=-[T^3X]-(\alpha-1)[T^2UV]-(\alpha+1)[UVY^2]+[UY^3]=-(T^3X)_{15}-(\alpha-1)(T^2UV)_{15}-(\alpha+1)(UVY^2)_8+[(Y^2)g_5]$

This takes care of terms involving g_6.

32. $(U)g_7-(Z)g_8=\alpha[UVX]=\alpha(UVX)_9$
33. $(Y)g_7-(Z)g_9=\alpha[VXY]=\alpha(VXY)_{18}$
34. $(X)g_7-(Z)g_{10}=[\alpha VX^2+TXZ]+[TVZ]=[(-T)g_7+(\alpha V)g_{10}+(\alpha T)g_{15}]+[(V)g_{11}+(-V)g_{13}+(T-(\alpha+1)X)g_{15}]$
35. $(T)g_7-(X)g_{11}=\alpha TVX-TXY-\alpha VX^2=\alpha[TVX-VX^2]-[TXY]=\alpha[(-V)g_{10}+(T)g_{15}]-(TXY)_6$
36. $(TY)g_7-(XZ)g_{13}=\alpha[TVXY]+[VX^2Z-TVXZ]=\alpha[(TV)g_9]+[(VZ)g_{10}+(TZ)g_{15}]$
37. $(T^2)g_7-(XZ)g_{14}=\alpha[T^2VX]-(\alpha+1)[UVXZ]+[VXYZ]=\alpha[(\alpha+1)(-V^2)g_8+(V^2)g_9+(VX)g_{14}]$

This takes care of terms involving g_7.

38. $(X)g_8-(U)g_{10}=[TUX]+[TUV]=(TUX)_5+(TUV)_{12}$

This takes care of terms involving g_8.

39. $(X)g_9-(Y)g_{10}=[TXY]+[TVY]=(TXY)_6+(TVY)_{21}$
40. $(T)g_9-(X)g_{13}=[VX^2-TVX]=[(V)g_{10}+(T)g_{15}]$

This takes care of terms involving g_9.

41. $(TZ)g_{10}-(X^2)g_{11}=-[T^2XZ]-[T^2VZ]-[TX^2Y]-\alpha[VX^3]=-(T^2XZ)_{29}-[(VZ)g_14+(YZ-(\alpha+1)UZ)g_{15}]-[(TX)g_9]-\alpha[(\alpha+1)(-V^2)g_8+(V^2)g_9+(VX+TV)g_{10}+(VX)g_{14}+(TX+T^2)g_{15}]$

42. $(TY)g_{10}-(X^2)g_{13} = -[T^2XY]-[T^2VY]+[VX^3]-[TVX^2] = -(T^2XY)_{14}-[(VY)g_{14}+(Y^2-(\alpha+1)UY)g_{15}]+(VX^3)_{41}-[(\alpha+1)(-V^2)g_8+(V^2)g_9+(TV)g_{10}+(VX)g_{14}+(T^2)g_{15}]$
43. $(T^2)g_{10}-(X^2)g_{14} = -[T^3X]-[T^3V]-(\alpha+1)[UVX^2]+[VX^2Y] = -(T^3X)_{15}-[(\alpha+1)(-V^2)g_{12}+(TV)g_{14}+(TY)g_{15}]-(\alpha+1)(UVX^2)_{11}+(VX^2Y)_{20}$

This takes care of terms involving g_{10}.

44. $(U)g_{11}-(Z)g_{12} = [TUY]+\alpha[UVX] = (TUY)_{12}+\alpha(UVX)_9$
45. $(Y)g_{11}-(Z)g_{13} = [TY^2]+\alpha[VXY]+[VXZ]-[TVZ] = (TY^2)_{21}+\alpha(VXY)_{18}+(VXZ)_3-(TVZ)_{22}$
46. $(T)g_{11}-(Z)g_{14} = [T^2Y+\alpha TVX-(\alpha+1)UVZ]+[VYZ] = [(\alpha+1)(-V)g_2+(\alpha+1)(-V)g_5+(V)g_6+(Y)g_{14}+(\alpha^2+1)(-U)g_{15}$

This takes care of terms involving g_{11}.

47. $(Y)g_{12}-(U)g_{13} = [UVX]-[TUV] = (UVX)_9-(TUV)_{12}$
48. $(T)g_{12}-(U)g_{14} = -(\alpha+1)[U^2V]+[UVY] = -(\alpha+1)(U^2V)_5+(UVY)_6$

This takes care of terms involving g_{12}.

49. $(T)g_{13}-(Y)g_{14} = -TVX+T^2V-(\alpha+1)UVY+VY^2 = [VY^2-TVX]+[T^2V]-(\alpha+1)[UVY] = [(V)g_6+(\alpha-1)(U)g_{15}]+[(V)g_{14}+(Y-(\alpha+1)U)g_{15}]-(\alpha+1)(UVY)_6$

This takes care of terms involving g_{13}, g_{14} and g_{15}, and thus completes all of the necessary computations!

References

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