On the growth of the Bass sequence of a Cohen-Macaulay local ring

(joint with G. Leuschke)

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R is a local ring with maximal ideal $\mathfrak{m}$ and residue field k.

Given a finitely generated R-module M, the *Betti numbers* of M are

$$\beta_i^R(M) = \dim_k \operatorname{Ext}_R^i(M, k)$$

$\{\beta_i^R(M)\}_{i\geq 0}$ is the *Betti sequence* and the *Poincaré series* is the formal power series

$$P_M^R(t) = \sum_{i\geq 0} \beta_i^R(M) t^i$$

The *Bass numbers* of M are

$$\mu_R^i(M) = \dim_k \operatorname{Ext}_R^i(k, M)$$

$\{\mu_R^i(M)\}_{i\geq 0}$ is the *Bass sequence* and the *Bass series* is the formal power series

$$B_R^M(t) = \sum_{i\geq 0} \mu_R^i(M) t^i$$

Problem: Describe the growth of the Bass sequence

$$\{\mu_R^i(R)\}_{i\geq 0}$$

of R.

Motivation:

Theorem. (Auslander-Buchsbaum-Serre, Gulliksen)

1. R is regular $\Longleftrightarrow$ $\mathrm{pd}_R\, k < \infty$

2. R is a ci $\Longleftrightarrow$ $\mathrm{cx}_R\, k < \infty$

ci = complete intersection

cx = complexity

$\mathsf{cx}_R\, M = d$ means

$$\beta_i^R(M) \leq p(i) \text{ for all } i \gg 0,$$

where p is a polynomial of degree $d-1$.

$\mathsf{px}_R\, M = d$ means

$$\mu_R^i(M) \leq p(i) \text{ for all } i \gg 0,$$

where p is a polynomial of degree $d-1$.

Thus

$$\mathsf{cx}_R\, M = 0 \Longleftrightarrow \mathsf{pd}_R\, M < \infty$$

and

$$\mathsf{px}_R\, M = 0 \Longleftrightarrow \mathsf{id}_R\, M < \infty$$

Theorem. (Auslander-Buchsbaum-Serre, Gulliksen)

1. R is regular $\iff \operatorname{pd}_R k < \infty$

2. R is a ci $\iff \operatorname{cx}_R k < \infty$

Theorem ???

1. R is Gorenstein $\iff \operatorname{id}_R R < \infty$

2. R is a "nice" $\iff \operatorname{px}_R R < \infty$

Assume R is Cohen-Macaulay with canonical module ω.

Can assume further that R has dimension zero. Then

$$\mu_R^i(R) = \beta_i^R(\omega)$$

and we have

Question 1. Does $\mathrm{cx}_R\,\omega < \infty$ imply that R is Gorenstein?

What is known.

Say that M (or $\{\beta_i^R(M)\}_{i\geq 0}$) has *exponential growth* if there exists a real number $a > 1$ such that $\beta_i^R(M) \geq a^i$ for all $i \gg 0$.

Assume that R is not Gorenstein. Then ω has exponential growth if

1. $\operatorname{codim} R \leq 3$

2. $\mathfrak{m}^3 = 0$

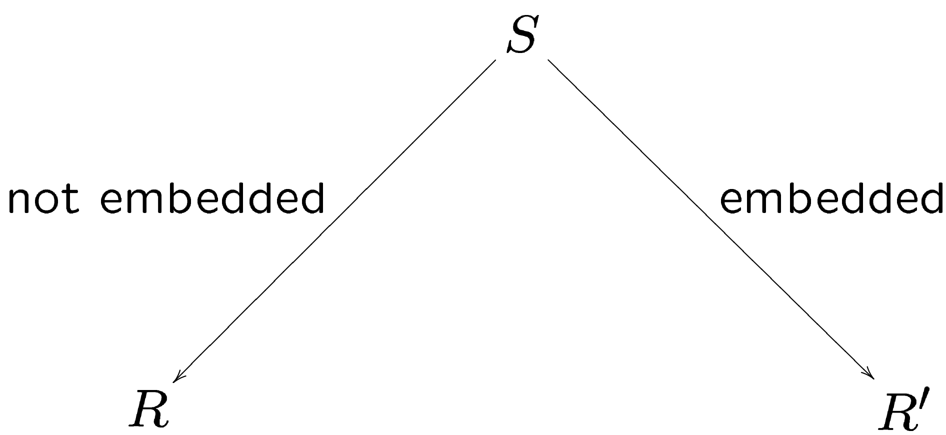
3. R is Golod

What makes this question difficult?

If $R = S/(f)$, f a nonzerodivisor, then $\omega_R \cong \omega_S / f\,\omega_S$, and ω_R and ω_S have the same growth.

If $R \to S$ is flat and $S/\mathfrak{m}S$ is Gorenstein, then $\omega_S = \omega_R \otimes_R S$, and ω_R and ω_S have the same growth.

Example 1.



In fact, if $\mathfrak{m}^3 = 0$, or R is Golod, then ω has *extremal growth*, meaning that

$$\operatorname{curv}_R(\omega) = \operatorname{curv}_R(k)$$

where the *curvature* of M is

$$\operatorname{curv}_R(M) = \limsup_{n\to\infty} \sqrt[n]{\beta_n(M)}$$

Question 2. Does ω always have extremal growth when R is not Gorenstein?

"Yes" to Question 2 implies "Yes" to question 1.

But the answer to Question 2 is "No."

Construction. R_1, R_2 essentially of finite type over the same field k, with k their common residue field. Let R be the localization of $R_1 \otimes_k R_2$ at the maximal ideal $\mathfrak{m} := \mathfrak{m}_1 \otimes_k R_2 + R_1 \otimes_k \mathfrak{m}_2$. If R_1 and R_2 are Cohen-Macaulay rings with canonical modules ω_1 and ω_2, then R is Cohen-Macaulay with canonical module

$$\omega_R = (\omega_1 \otimes_k \omega_2)_{\mathfrak{m}}.$$

Let M_1 and M_2 be finitely generated modules over R_1 and R_2, respectively. Put $M = (M_1 \otimes_k M_2)_{\mathfrak{m}}$. Then we have an equality of Poincaré series

$$P_M^R(t) = P_{M_1}^{R_1}(t) P_{M_2}^{R_2}(t) \,.$$

For $M = (M_1 \otimes_k M_2)_{\mathfrak{m}}$ we have

$$\operatorname{curv}_R(M) = \max\{\operatorname{curv}_{R_1}(M_1), \operatorname{curv}_{R_2}(M_2)\} \,.$$

Example 2. Take Gorenstein R_1 with large $\mathrm{curv}_{R_1}(k)$, and non-Gorenstein R_2 with both $\mathrm{curv}_{R_2}(k)$ and $\mathrm{curv}_{R_2}(\omega_2)$ small:

$$R_1 = k[x_1, \dots, x_e]/I$$

where

$$I = (x_i^2 - x_{i+1}^2, x_j x_l \mid i = 1, \dots, e-1; j \neq l)$$

$e \geq 3$. Then R_1 is a Gorenstein ring, and $\mathrm{curv}_{R_1}(k) = \frac{2}{e-\sqrt{e^2-4}}$. Note that $\frac{2}{e-\sqrt{e^2-4}} \to \infty$ as $e \to \infty$.

$$R_2 = k[a,b]/(a^2, ab, b^2)$$

The curvature of every nonfree finitely generated R_2-module is 2.

Now let R be the tensor of R_1 and R_2, localized. Then

$$\mathrm{curv}_R(\omega_R) = 2 < \frac{2}{e - \sqrt{e^2 - 4}} = \mathrm{curv}_R(k)$$

Note: The previous construction also shows that ω is not in general a test module for finite projective dimension:

Let M_1 be any R_1-module with $\operatorname{pd}_{R_1} M_1 = \infty$. Then $\operatorname{pd}_R M = \infty$ where $M = (M_1 \otimes_k R_2)_{\mathfrak{m}}$, and

$$\operatorname{Tor}_i^R(M, \omega) = 0 \text{ for all } i > 0$$

Note: The canonical module ω does not have *finite virtual projective dimension*.

Question 3. Can the canonical module have *finite complete intersection dimension*?

Theorem. Let R be a standard graded ring over a field k. Assume that for linearly independent elements $x, y \in R_1$,

$$x^2 = 0 \quad \text{and} \quad xy = 0.$$

If either

1. $y^2 = 0$

or

2. R is a quotient by monomials

then the canonical module has exponential growth.